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SOME ASSOCIATION BETWEEN R_α, R_β AND R_m IN TERMS OF THETA FUNCTIONS

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Abstract. The authors establish a set of nine new theta-function identities involving R_α, R_β and R_m, $m \in \mathbb{Z}^+$ functions, which are based upon a number of q-product identities and Jacobi's celebrated triple-product identity. These theta function identities depict the interrelationships that exist among theta function identities and combinatorial partition-theoretic identities.

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1. Introduction

Throughout this article, we employ the notation

$$(\lambda; q)_\infty := \prod_{i=0}^{\infty}\left(\frac{1-\lambda q^i}{1-\lambda q^{m+i}}\right)$$

for any real or complex numbers q, λ and m with $|q| < 1$ so that

$$(\lambda; q)_n := \begin{cases} 1 & (n = 0) \\ (1-\lambda)(1-\lambda q)(1-\lambda q^2)\dots(1-\lambda q^{n-1}) & (n \in \mathbb{N}) \end{cases}$$

and

$$(\lambda; q)_\infty = \lim_{n\to\infty}(\lambda; q)_n = \prod_{i=0}^{\infty}(1-\lambda q^i).$$

Also, for convenience we write

$$(\lambda_1, \lambda_2, \dots, \lambda_n; q)_\infty = (\lambda_1; q)_\infty(\lambda_2; q)_\infty \dots (\lambda_n; q)_\infty.$$

Ramanujan [4, p. 31, Eq. (18.1)] defined the general theta function $f(a, b)$ as follows:

$$f(a, b) := 1 + \sum_{n=1}^{\infty}(ab)^{n(n-1)/2}(a^n + b^n), \qquad |ab| < 1.$$

The above identity enjoys the famous Jacobi's triple product identity [4, p. 35, Entry 19]

$$f(a, b) := (-a; ab)_\infty(-b; ab)_\infty(ab; ab)_\infty.$$

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The three important q-series identities, which emerge naturally from the above two identities and are worth noting here.

$$\varphi(q) := \mathfrak{f}(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty = \frac{(-q;q^2)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty},$$

$$\psi(q) := \mathfrak{f}(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

and

$$f(-q) := \mathfrak{f}(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.$$

By introducing the general family $R(s,t,l,u,v,w)$, Andrews et al. [2] found a number of interesting double summation hypergeometric q-series representation for several families of partitions and explored the role of double series in combinatorial-partition identities:

$$R(s,t,l,u,v,w) := \sum_{n=0}^{\infty} q^{s\binom{n}{2}+tn}\ r(l,u,v,w;n), \tag{1}$$

where

$$r(l,u,v,w;n) := \sum_{i=0}^{[\frac{n}{u}]} \frac{(-1)^i\ q^{uv\binom{i}{2}+(w-ul)i}}{(q;q)_{n-ui}\ (q^{uv};q^{uv})_i}.$$

We also recall the following interesting special cases of (1) (see, for details,[2, p. 106, Thorem 3]; see also [16]. Also, recently Srivastava et al. [17] represented them in terms of the notations R_α, R_β and R_m, $m \in \mathbb{Z}^+$.

$$R_\alpha := R(2,1,1,1,2,2) = (-q;q^2)_\infty, \tag{2}$$

$$R_\beta := R(2,2,1,1,2,2) = (-q^2;q^2)_\infty, \tag{3}$$

and

$$R_m := R(m,m,1,1,1,2) = \frac{(q^{2m};q^{2m})_\infty}{(q^m;q^{2m})_\infty}. \tag{4}$$

Further, several new advancements and generalizations of existing results were made in regard to combinatorial partition-theoretic identities. For the wonderful work one may refer [6, 7, 8, 9, 11]. An interesting recent investigation on the subject of combinatorial partition-theoretic identities by Hahn et al. [12] is also worth mentioning in this connection.

Motivated by the above work, in this paper we establish many new theta-function identities which depict the inter-relationships in terms of R_α, R_β and R_m, $m \in \mathbb{Z}^+$ functions along with q-product identities.

2. Preliminaries

In this section, we list some preliminary results, which we need to prove our main results.

Lemma 2.1. [13, Theorem 3.1(i)] *If*

$$M=\frac{f_1}{q^{1/24}f_2} \text{ and } N=M(q^3)$$

then, we have

$$(MN)^3+\frac{8}{(MN)^3}=\left(\frac{N}{M}\right)^6-\left(\frac{M}{N}\right)^6.$$

Lemma 2.2. [15, Theorem 3.7] *If*

$$M=\frac{f_1}{q^{1/24}f_2} \text{ and } N=M(q^5)$$

then, we have

$$(MN)^2+\frac{4}{(MN)^2}=\left(\frac{N}{M}\right)^3-\left(\frac{M}{N}\right)^3.$$

Lemma 2.3. [3, p. 36,Theorem 3.5.1] *If*

$$M=\frac{f_1}{q^{1/8}f_4} \text{ and } N=M(q^2)$$

then, we have

$$(MN)^4+\frac{256}{(MN)^4}=\left(\frac{N}{M}\right)^{12}-16\left(\frac{N}{M}\right)^4-16\left(\frac{M}{N}\right)^4.$$

Lemma 2.4. [3, Theorem 2.3] *If*

$$M=\frac{f_1}{q^{1/8}f_4} \text{ and } N=M(q^3)$$

then, we have

$$MN+\frac{4}{MN}=\left(\frac{M}{N}\right)^2+\left(\frac{N}{M}\right)^2.$$

Lemma 2.5. [5, Entry 53 p. 206] [14, p. 325] *If*

$$M=\frac{f_1}{q^{1/6}f_5} \text{ and } N=M(q^2)$$

then, we have

$$MN+\frac{5}{MN}=\left(\frac{M}{N}\right)^3+\left(\frac{N}{M}\right)^3.$$

Lemma 2.6. [5, Entry 55 p. 209] *If*

$$M=\frac{f_1^2}{q^{1/2}f_7^2} \text{ and } N=M(q^2)$$

then, we have

$$MN+\frac{49}{MN}=\left(\frac{N}{M}\right)^3-8\frac{N}{M}-8\frac{M}{N}+\left(\frac{M}{N}\right)^3.$$

Lemma 2.7. [1, Theorem 5.3] *If*

$$M=\frac{\psi(-q)}{q^{1/2}\psi(-q^5)}\ \text{and}\ N=\frac{\varphi(q)}{\varphi(q^5)}$$

then, we have

$$N^2+M^2N^2=5+M^2.$$

Lemma 2.8. [1, Theorem 5.2] *If*

$$M=\frac{\psi(-q)}{q\psi(-q^9)}\ \text{and}\ N=\frac{\varphi(q)}{\varphi(q^9)}$$

then, we have

$$N+MN=3+M.$$

Lemma 2.9. [5, Entry 65 p. 230] *If*

$$M=\frac{f_3f_5}{q^{1/3}f_1f_{15}}\ \text{and}\ N=M(q^2)$$

then, we have

$$MN+\frac{1}{MN}=\left(\frac{N}{M}\right)^3+\left(\frac{M}{N}\right)^3+4.$$

3. Main Results

Theorem 3.1. *Each of the following relationship holds true.*

$$\left\{\frac{(q,q^2,q^3,q^3;q^3)_\infty}{(q^2,q^4,q^6,q^6;q^6)_\infty}\right\}^3+8q\left\{\frac{(q^2,q^4,q^6,q^6;q^6)_\infty}{(q,q^2,q^3,q^3;q^3)_\infty}\right\}^3$$

$$=\left\{\frac{R_1(q^6;q^6)_\infty}{R_3(q^2;q^2)_\infty}\right\}^6-q\left\{\frac{R_3(q^2;q^2)_\infty}{R_1(q^6;q^6)_\infty}\right\}^6, \tag{5}$$

which gives the inter-relationship between R_1 and R_3.

$$\left\{\frac{(q,q^2,q^3,q^4,q^5,q^5;q^5)_\infty}{(q^2,q^4,q^6,q^8,q^{10},q^{10};q^{10})_\infty}\right\}^2+4\left\{\frac{(q^2,q^4,q^6,q^8,q^{10},q^{10};q^{10})_\infty}{(q,q^2,q^3,q^4,q^5,q^5;q^5)_\infty}\right\}^2$$

$$=q^{1/6}\left\{\frac{R_1(q^{10};q^{10})_\infty}{R_5(q^2;q^2)_\infty}\right\}^3-q^{1/6}\left\{\frac{R_5(q^2;q^2)_\infty}{R_1(q^{10};q^{10})_\infty}\right\}^3, \tag{6}$$

which gives the inter-relationship between R_1 and R_5.

$$\left\{\frac{(q,q^2,q^2;q^2)_\infty}{(q^4,q^8,q^8;q^6)_\infty}\right\}^4+256q^3\left\{\frac{(q^4,q^8,q^8;q^6)_\infty}{(q,q^2,q^2;q^2)_\infty}\right\}^4$$

$$=\left\{\frac{R_1(q^8;q^8)_\infty}{R_4(q^2;q^2)_\infty}\right\}^{12}-16q^2\left\{\frac{R_1(q^8;q^8)_\infty}{R_4(q^2;q^2)_\infty}\right\}^4-16q^2\left\{\frac{R_4(q^2;q^2)_\infty}{R_1(q^8;q^8)_\infty}\right\}^4, \tag{7}$$

which gives the inter-relationship between R_1 and R_4.

$$\frac{(q,q^2,q^3,q^3;q^3)_\infty}{(q^4,q^8,q^{12},q^{12};q^{12})_\infty}+\frac{4q(q^4,q^8,q^{12},q^{12};q^{12})_\infty}{(q,q^2,q^3,q^3;q^3)_\infty}$$

$$=q\left\{\frac{R_6(q;q^2)_\infty(q^4;q^4)_\infty}{R_2(q^3;q^6)_\infty(q^{12};q^{12})_\infty}\right\}^2+\left\{\frac{R_2(q^3;q^6)_\infty(q^{12};q^{12})_\infty}{R_6(q;q^2)_\infty(q^4;q^4)_\infty}\right\}^2, \tag{8}$$

which gives inter-relationship between R_2 and R_6.

Proof of (6). Rewriting M and N in terms of bases q^5, q^{10} in Lemma 2.2 and employing (4), we obtain

$$MN = q^{16}\frac{(q,q^2,q^3,q^4,q^5,q^5;q^5)_\infty}{(q^2,q^4,q^6,q^8,q^{10},q^{10};q^{10})_\infty}$$

and

$$\frac{M}{N} = q^{1/6}\frac{R_5(q^2;q^2)_\infty}{R_1(q^{10};q^{10})_\infty}.$$

On employing MN and M/N in Lemma 2.2, we complete the proof. □

Proof of (7). Rewriting M and N in terms of bases q^2, q^8 in Lemma 2.3 and employing (4), we obtain

$$MN = \frac{1}{q^{3/8}}\frac{(q,q^2,q^2;q^2)_\infty}{(q^4,q^8,q^8;q^6)_\infty}$$

and

$$\frac{M}{N} = q^{1/8}\frac{R_4(q^2;q^2)_\infty}{R_1(q^8;q^8)_\infty}.$$

On employing MN and M/N in Lemma 2.3, we complete the proof. □

Proof of (8). Rewriting M and N in terms of bases q^3, q^{12} in Lemma 2.4 and employing (4), we obtain

$$MN = \frac{1}{q^{1/2}}\frac{(q,q^2,q^3,q^3;q^3)_\infty}{(q^4,q^6,q^{12},q^{12};q^{12})_\infty}$$

and

$$\frac{M}{N} = q^{1/4}\frac{R_6(q;q^2)_\infty(q^4;q^4)_\infty}{R_2(q^3;q^6)_\infty(q^{12};q^{12})_\infty}.$$

On employing MN and M/N in Lemma 2.4, we complete the proof. □

Proof of (9). Rewriting M and N in terms of bases q^2, q^{10} in Lemma 2.5 and employing (4), we obtain

$$MN = \frac{1}{q^{1/2}}\frac{(q,q^2,q^2;q^2)_\infty}{(q^5,q^{10},q^{10};q^{10})_\infty}$$

and

$$\frac{M}{N} = q^{1/6}\frac{R_5(q^2;q^2)_\infty}{R_1(q^{10};q^{10})_\infty}.$$

On employing MN and M/N in Lemma 2.5, we complete the proof. □

Proof of (10). Rewriting M and N in terms of bases q^2, q^{14} in Lemma 2.6 and employing (4), we obtain

$$MN = \frac{1}{q^{3/2}}\frac{(q,q,q^2,q^2,q^2,q^2;q^2)_\infty}{(q^7,q^7,q^{14},q^{14},q^{14},q^{14};q^{14})_\infty}$$

and

$$\frac{M}{N} = q^{1/2}\frac{R_7^2(q^2,q^2;q^2)_\infty}{R_1^2(q^{14},q^{14};q^{14})_\infty}.$$

On employing MN and M/N in Lemma 2.6, we complete the proof. □

Proof of (11). Replacing $q \to -q$ in Lemma 2.7 and rewriting M and N in terms of bases q^2 and q^{10} and employing (2)–(4), we obtain

$$M = \frac{R_1}{q^{1/2}R_5} \text{ and } N = \frac{(q, q^2; q^2)_\infty(-q^5, -q^{10}; q^{10})_\infty}{R_\alpha R_\beta (q^5, q^{10}; q^{10})_\infty}.$$

On employing M and N in Lemma 2.7, we complete the proof. □

Proof of (12). Replacing $q \to -q$ in Lemma 2.8 and rewriting M and N in terms of bases q^2 and q^{18} and employing (2)–(4), we obtain

$$M = -\frac{1}{q}\frac{R_1}{R_9} \text{ and } N = \frac{(q, q^2; q^2)_\infty(-q^9, -q^{18}; q^{18})_\infty}{R_\alpha R_\beta (q^9, q^{18}; q^{18})_\infty}.$$

On employing M and N in Lemma 2.8, we complete the proof. □

Proof of (13). Rewriting M and N in terms of bases q^2, q^6, q^{10} and q^{10} in Lemma 2.9 and employing (4), we obtain

$$MN = \frac{(q^3, q^6, q^6; q^6)_\infty(q^5, q^{10}, q^{10}; q^{10})_\infty}{q(q, q^2, q^2; q^2)_\infty(q^{15}, q^{30}, q^{30}; q^{30})_\infty}$$

and

$$\frac{M}{N} = q^{1/3}\frac{R_1\ R_{15}\ (q^6; q^6)_\infty(q^{10}; q^{10})_\infty}{R_3\ R_5(q^2; q^2)_\infty(q^{30}; q^{30})_\infty}.$$

On employing MN and M/N in Lemma 2.9, we complete the proof. □

4. Concluding Remarks and Observations

The present investigation was motivated by several recent developments dealing essentially with theta-function identities and combinatorial partition-theoretic identities. Here, in this article, we have established nine presumably new theta-function identities which depict the inter-relationships that exist among between R_α, R_β and R_m and combinatorial partition-theoretic identities. In particular, the recent works by Chaudhary (see [7] - [9]), Chaudhary *et al.* (see [10] -[11]), and Srivastava *et al.* (see [18]) are worth mentioning here.

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